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Quasi-steady flames on an evolving atmosphere

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Summary

By setting up equations for the differences between local values of temperature and concentration, and values appropriate to a spatially uniform, but temporally evolving, reactive atmosphere, the propagation of a plane quasi-steady combustion wave (flame) can be studied in an unbounded atmosphere. Confining attention (arbitrarily) at this stage to waves of low speed, comparable with generally accepted thermal flame-speed values, it is found that a continuous spectrum of quasi-steady wave speeds is possible. The structures of these waves are dependent on their speed, and are similar to the ones recently found for half-space burner flames. The actual speed of wave propagation in any particular case must depend upon details of the larger disturbance field into which the quasi-steady flame must fit, with obvious implications for flame acceleration and related matters.

0. Introduction

Theoretical studies of pre-mixed flames are complicated by the fact that the atmosphere ahead of a combustion wave is necessarily in a state of chemical disequilibrium. Various tricks or devices have been proposed to overcome this "cold-boundary difficulty" (see e.g. [11, Ch. 5, Section 4b]; Buckmaster and Ludford [2, especially Ch. 2] exploit asymptotic analysis based on the notion of a large activation energy for the reaction and therefore disregard exponentially small reaction rates) but they are not wholly satisfactory, particularly from the point of view of reaction kinetics. For example, the Arrhenius exponential term is sometimes modified to read $\exp[-E'_A/R'(T'-T'_0)]$ instead of $\exp(-E'_A/R'T')$, where T'_0 is the ambient-atmosphere value of the absolute temperature T', E'_A is the activation energy per mole and R' is the universal gas constant; the reaction is therefore artificially switched off when T' is equal to T'_0.

Since $E'_A/R'T'$ is usually a large number, the reaction rate under ambient atmospheric conditions is exceptionally small and one intuitively feels that it must be of negligible practical significance. Thus the asymptotic approach that simply discards terms of exponentially small size is attractive from the point of view of intuitive physics, as well as for the rather more solid reasons that many of its predictions accord so well with observations of thermal-flame structure and behaviour.

The avoidance of "cold-boundary" problems by adoption of half-space flameholder [7] models of plane steady combustion wave behaviour is a step that has its attractions, since it only requires some assertion about the flux of the products of combustion at the flameholder (usually that the flux is zero) in order to complete formulation of the problem. The concession that this demands from a rational model of the physics of the

situation is slight, and the configuration has recently been exploited to study plane combustion waves whose speeds (i.e., flow speed through the face of the holder) vary continuously from conventional thermal-flame values, through a transitional range of structures, to "convected-explosion flames" at the threshold of the appearance of compressibility effects [3] and on into the regimes of fully-compressible reactive gas flow [4,10], These studies make it quite clear that chemical reaction rates are of first-order importance in domains other than the one prescribed by classical thermal-flame structure, namely the mainly diffusion-reaction-structured flame sheet that is found downstream of a predominantly inert convection-diffusion-balanced pre-heat zone, even though the rates in these domains are exponentially smaller than flame-sheet values.

The existence of a thermal flame at the far-downstream tail of a long precursor (convected-explosion) region of relatively small temperature variations means that the half-space model can be exploited to study other aspects of flame behaviour than the simply structural but, despite this fact, there is a persistent feeling that it is both necessary and possible to model combustion waves in essentially unbounded domains. If the lessons that have been learnt from the flameholder models are to be heeded one must not alter Arrhenius reaction kinetics in any way, which means that one must necessarily accept that the ambient atmosphere into which the combustion wave is propagating is not in a steady state.

The only solution of the conservation equations that is devoid of all spatial non-uniformity demands both zero speed of gas movement and constant density; pressure and temperature T_0' will then both be increasing with time. Having first established a suitable set of basic equations, the present analysis sets out to investigate the behaviour of the *difference* $(T' - T_0')$ when this difference is representative of a propagating combustion wave. It follows that $(T' - T_0')$ will be driven by a reaction rate that is the excess of the local rate over the ambient constant-volume rate appropriate to T_0'.

A prime assumption, made early on in the analysis, is that, relative to the combustion wave, the field is quasi-steady. To this extent the present ideas are quite like the ones described by Aldushin et al. [1]. (This paper was brought to the author's attention by Professor Paul Clavin after the present analysis had been completed; Dr. M. Nettleton kindly made a reprint available to the author.) However, there are important points of difference between the two analyses; in particular there is the assumption by Aldushin et al. that the basic equation is a reaction-diffusion equation without convection (change of local convection rates, or gas-speeds, is an important feature of all real combustion waves), and its subsequent modification to describe the field of a temperature ratio (like T'/T_0') rather than a difference $(T' - T_0')$. As a consequence, there is an implication in Aldushin et al's analysis that the reaction-rate difference that appears in their equation for T'/T_0' is a difference between local and far-upstream reaction rates, *both* taken at constant pressure. In broad terms, and certainly for the purpose of their investigation, this matter is not of great significance to the work of Aldushin et al., but in application to other problems than theirs this difference can be significant. There can be no doubt that the present analysis of the temperature difference $(T' - T_0')$ is firmly based on the proper conservation equations and their consequences.

An extensive, basic, and most interesting analysis of the structure and speed of propagation of pre-mixed flames, that makes use of the model of Aldushin et al. has been undertaken recently by Clavin and Liñán [6]. To the extent that phase-plane pictures like those in Fig. 1 here also appear in Clavin's and Liñán's paper there are evidently similarities between their work and the present study. However, one must reiterate that the

Aldushin et al. model examines T'/T_0' whereas here we deal with $(T' - T_0')$; also Clavin and Liñán are principally interested in questions of uniqueness and, especially, for a certain class of reaction-rate models, of the changes in reaction-wave propagation speed with (effectively) activation energy of the reaction, starting from zero. Thus the present work complements the study by Clavin and Liñán and, most particularly, indicates how matters go when reaction-wave speeds are in excess of the conventional adiabatic thermal-flame speeds. The length scales of these faster flames can be realistically short and "observable" when their ambient atmospheres are hot, and they may well therefore be key elements in transitional or unsteady combustion processes.

After listing the basic conservation and other equations in Section 1, Section 2 describes the main features of the constant-density ambient atmosphere. These results are combined in Section 3 to produce general and exact equations for differences in stagnation temperature and reactant concentration. With the assumption of quasi-steady conditions and very small Mach number of wave propagation, a single equation for the temperature difference is derived, and its general solution and consequences are described in Sections 4 and 5.

1. Basic equations

The energy, species and momentum equations for a one-dimensional unsteady flow are, respectively,

$$\rho' C_p' \left\{ \frac{\partial T_s'}{\partial t'} + u' \frac{\partial T_s'}{\partial x'} \right\} = \frac{\partial p'}{\partial t'} + \frac{\partial}{\partial x'} \left(\lambda' \frac{\partial T'}{\partial x'} \right) + \frac{\partial}{\partial x'} \left\{ \tfrac{4}{3} \, \mathrm{Pr} \frac{\lambda'}{C_p'} \frac{\partial}{\partial x'} \left(\tfrac{1}{2} u'^2 \right) \right\} + \rho' W_P Q' \mathscr{R}', \tag{1.1}$$

$$\rho' \left\{ \frac{\partial c_\alpha}{\partial t'} + u' \frac{\partial c_\alpha}{\partial x'} \right\} = \frac{\partial}{\partial x'} \left\{ \mathrm{Le} \frac{\lambda'}{C_p'} \frac{\partial c_\alpha}{\partial x'} \right\} - \rho' W_\alpha \mathscr{R}', \quad \alpha = \mathrm{X}, \mathrm{F}, \tag{1.2}$$

$$\rho' \left\{ \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{k x'} \right\} = - \frac{\partial p'}{\partial x'} + \frac{\partial}{\partial x'} \left\{ \tfrac{4}{3} \, \mathrm{Pr} \frac{\lambda'}{C_p'} \frac{\partial u'}{\partial x'} \right\}, \tag{1.3}$$

where p', ρ', T', T_s', u', c_α are pressure, density, temperature, stagnation temperature, that is,

$$T_s' = T' + \frac{1}{2} \frac{u'^2}{C_p'}, \tag{1.4}$$

velocity and mass fraction of species α ($= \mathrm{X}$, F for oxidant and fuel, respectively). Thermal conductivity is written as λ' while Pr and Le are the Prandtl and Lewis numbers. The specific heat at constant pressure is C_p' (assumed constant), Q' is the combustion energy per unit mass, W_α is the molecular weight of species α; $\alpha = \mathrm{P}$ signifies the product species and

$$W_P = W_X + W_F \equiv W_F (1 + \sigma). \tag{1.5}$$

By implication the combustion reaction is

$$F + X \rightarrow P,$$

whose rate of progress is

$$\mathscr{R} = \frac{1}{\tau'_{ch}} c_X c_F. \tag{1.6}$$

The chemical time τ'_{ch} is dependent upon activation energy E'_A, sound speed a'_f, etc, as follows:

$$\frac{1}{\tau'_{ch}} = \frac{A_3}{W_F} \frac{a'^2_f}{\mathscr{K}' \, Pr} e^{-E'_A/R'T'} \left(\frac{E'_A}{R'T'} \right)^s, \tag{1.7}$$

where R' is the universal gas constant and $\mathscr{K}'$ is the thermal diffusivity,

$$\mathscr{K}' = \lambda'/\rho' C'_p \tag{1.8}$$

The index s in (1.7) is a small number, of order ± 1 for example, while A_3 is also a number of order unity, but essentially positive.

The thermal equation of state will be used in the form

$$p' = \rho' C'_p \left(\frac{\gamma_f - 1}{\gamma_f} \right) T', \tag{1.9}$$

where γ_f is the (constant) ratio of frozen specific heats, and the continuity equation is

$$\frac{\partial \rho'}{\partial t'} + \frac{\partial}{\partial x'} (\rho' u') = 0. \tag{1.10}$$

The frozen sound speed is a'_f, such that

$$a'^2_f = \gamma_f p'/\rho' = C'_p (\gamma_f - 1) T', \tag{1.11}$$

and this will be useful later on.

2. Ambient atmosphere

If a combustible atmosphere is spatially uniform, so that $\partial(\ )/\partial x'$ is zero for all $(\ )$, (1.1)–(1.3) give

$$\rho'_0 C'_p T'_{0t'} - p_{0t'} = \rho'_0 W_P Q' \mathscr{R}'_0, \tag{2.1}$$

$$c_{\alpha 0t'} = -W_\alpha \mathscr{R}'_0, \quad \alpha = X, F, \tag{2.2}$$

$$u'_{0t'} = 0 \Rightarrow u'_0 = 0 \quad \text{(say)}, \tag{2.3}$$

where subscript 0 denotes the value of a quantity in this atmosphere. Equation (1.10) shows that

$$\rho'_0 = \text{constant}, \tag{2.4}$$

whence (1.9) gives

$$p'_{0t'} = \rho'_0 C'_p \left(\frac{\gamma_f - 1}{\gamma_f} \right) T'_{ot'}. \tag{2.5}$$

Equations (2.1) and (2.5) combine to give

$$C'_v T'_{ot'} = W_P Q' \mathscr{R}'_0, \tag{2.6}$$

and this result together with (2.2) shows that

$$C'_v T'_0 + Q'(c_{X0} + c_{F0}) = \text{const.}, \tag{2.7}$$

where C'_v is specific heat at constant volume.

Similarly (2.2), with $\alpha = X, F$, give

$$\frac{c_{X0}}{W_X} - \frac{c_{F0}}{W_F} = \text{const.} = \frac{c_{X0i}}{W_X} - \frac{c_{F0i}}{W_F} \tag{2.8}$$

where subscript i signifies an initial value. Thus

$$c_{X0} = c_{X0i} + \sigma(c_{F0} - c_{F0i}), \tag{2.9}$$

and (2.7) can now be written as

$$C'_v T'_0 + (1 + \sigma) Q' c_{F0} = \text{constant}. \tag{2.10}$$

It will be assumed from now on that the fuel/oxidant mixture is a lean one, so that combustion ceases when c_F, or in the particular case c_{F0}, is zero. Thus (2.10) can be written as

$$C'_v (T'_{0f} - T'_0) = (1 + \sigma) Q' c_{F0}, \tag{2.11}$$

where T'_{0f} is the final temperature in the present constant-volume process.

In terms of initial values T'_{0i} and c_{F0i}, (2.10) can be written as

$$C'_v T'_0 + (1 + \sigma) Q' c_{F0} = C'_v T'_{0i} + (1 + \sigma) Q' c_{F0i}. \tag{2.12}$$

Thus

$$T'_{0f} = T'_{0i} + (1 + \sigma) \frac{Q'}{C'_v} c_{F0i}, \tag{2.13}$$

and T'_{0f} is a constant for any given initial conditions.

3. Derivation of an equation for the temperature

Since T'_0 and $c_{\alpha 0}$ are functions of t' only, (2.1) can be subtracted from (1.1), and (2.2)

from (1.2), to give the following equations for the differences $T'_s - T'_0$ and $c_F - c_{F0}$:

$$\rho' C'_p \left\{ \frac{\partial}{\partial t'}(T'_s - T'_0) + u' \frac{\partial}{\partial x'}(T'_s - T'_0) \right\}$$

$$= \frac{\partial p'}{\partial t'} - \frac{\rho'}{\rho'_0} \frac{\partial p'_0}{\partial t'} + \frac{\partial}{\partial x'} \left\{ \lambda' \frac{\partial}{\partial x'}(T'_s - T'_0) \right\} + \rho' Q' W_P (\mathscr{R}' - \mathscr{R}'_0), \tag{3.1}$$

$$\rho' \left\{ \frac{\partial}{\partial t'}(c_\alpha - c_{\alpha 0}) + u' \frac{\partial}{\partial x'}(c_\alpha - c_{\alpha 0}) \right\}$$

$$= \frac{\partial}{\partial x'} \left\{ \frac{\lambda'}{C'_p} \frac{\partial}{\partial x'}(c_\alpha - c_{\alpha 0}) \right\} - \rho' W_\alpha (\mathscr{R}' - \mathscr{R}'_0), \ \alpha = \mathrm{X}, \mathrm{F}. \tag{3.2}$$

Prandtl number has been given the value $\frac{3}{4}$ and Lewis number the value unity; these assumptions make the analysis simpler without impeding seriously the proper modelling of the events whose progress is to be followed. At this point the proposition will be adopted that these events are of a quasi-steady character so that, in a first approximation at least, the partial time derivatives of $T'_s - T'_0$ and $c_\alpha - c_{\alpha 0}$ can be neglected relative to each of the other terms in the equations, including the ones proportional to $\mathscr{R}'_0$. As with any proposal for an approximation it must be inspected subsequently to be sure of its validity, and this is done in Section 5 below. The implication is, in essence at this stage, that the typical time for changes in $(T'_s - T'_0)$ and $(c_\alpha - c_{\alpha 0})$ is longer than τ'_{ch} (see (1.6)) in the ambient atmosphere: there is no *a priori* reason why these *differences* should vary at the same rate as T'_0 or $c_{\alpha 0}$ separately.

By the same token (1.10) simplifies to the extent that

$$\rho' u' \simeq m'(t') \simeq \rho'_\infty u'_\infty, \tag{3.3}$$

where m' is a quasi-steady, or slowly-varying mass-flux and ρ'_∞, u'_∞ are the values of density and flow velocity where $x' \to -\infty$.

In terms of the variable ξ, where

$$\xi = \int_0^{x'} (m' C'_p / \lambda') \mathrm{d}x', \tag{3.4}$$

the quasi-steady version of (3.1) and (3.2) become

$$\frac{\partial}{\partial \xi}(T'_s - T'_0) - \frac{\partial^2}{\partial \xi^2}(T'_s - T'_0) \simeq (1 + \sigma) \frac{Q'}{C'_p} W_F \frac{\mathscr{K}'}{u'^2} (\mathscr{R}' - \mathscr{R}'_0), \tag{3.5}$$

$$\frac{\partial}{\partial \xi}(c_\alpha - c_{\alpha 0}) - \frac{\partial^2}{\partial \xi^2}(c_\alpha - c_{\alpha 0}) \simeq - W_\alpha \frac{\mathscr{K}'}{u'^2} (\mathscr{R}' - \mathscr{R}'_0). \tag{3.6}$$

Equations (1.5) and (1.8) have been used in the derivation of these results, which show that

$$T'_s - T'_0 + (1 + \sigma)(Q'/C'_p)(c_F - c_{F0}) \simeq \tfrac{1}{2}\left(u'^2_\infty(t')/C'_p\right), \tag{3.7}$$

since $T' \to T'_0$ and $c_F \to c_{F0}$ where $\xi \to -\infty$, by hypothesis.

With the lean-mixture assumption (1.6) and (1.7) can be used to write

$$(1+\sigma)\frac{Q'}{C'_p}(\mathscr{R}' - \mathscr{R}'_0) \simeq c_{X0i}\frac{4}{3}\frac{A_3}{W_F}\left\{\left(\frac{\theta'}{T'}\right)^s \frac{a'^2_f}{\mathscr{K}'} e^{-\theta'/T'} c_F (1+\sigma)\frac{Q'}{C'_p}\right.$$

$$\left. - \left(\frac{\theta'}{T'_0}\right)^s \frac{a'^2_{f0}}{\mathscr{K}'_0} e^{-\theta'/T'_0} c_{F0}(1+\sigma)\frac{Q'}{C'_p}\right\}, \tag{3.8}$$

where

$$\theta' \equiv E'_A/R'. \tag{3.9}$$

But (3.7) makes

$$(1+\sigma)Q'c_F = (1+\sigma)Q'c_{F0} - C'_p(T'_s - T'_0) + \tfrac{1}{2}u'^2_\infty, \tag{3.10}$$

or, in view of the fact that the reaction will cease when c_F vanishes and T'_s takes the value T'_{sb}, where T'_{sb} is given by

$$C'_p(T'_{sb} - T'_0) = (1+\sigma)Q'c_{F0} + \tfrac{1}{2}u'^2_\infty, \tag{3.11}$$

one can write (3.10) in the form

$$(1+\sigma)Q'c_F = C'_p(T'_{sb} - T'_s). \tag{3.12}$$

Equation (2.11) shows that

$$(1+\sigma)Q'c_{F0} = C'_v(T'_{0f} - T'_0), \tag{3.13}$$

so that (3.8) with (3.12) and (3.13) becomes

$$(1+\sigma)\frac{Q'}{C'_p}(\mathscr{R}' - \mathscr{R}'_0)$$

$$\simeq c_{X0i}\frac{4}{3}\frac{A_3}{W_F}\left\{\left(\frac{\theta'}{T'}\right)^s \frac{a'^2_f}{\mathscr{K}'} e^{-\theta'/T'}(T'_{sb} - T') - \left(\frac{\theta'}{T'_0}\right)^s \frac{a'^2_{f0}}{\mathscr{K}'_0} e^{-\theta'/T'_0}\frac{1}{\gamma_f}(T'_{0f} - T'_0)\right. \tag{3.14}$$

Combination of (3.5) and (3.14) gives

$$\frac{\partial}{\partial\xi}(T'_s - T'_0) - \frac{\partial^2}{\partial\xi^2}(T'_s - T'_0)$$

$$\simeq c_{X0i}\frac{4A_3}{3}\frac{a'^2_f}{u'^2}\left(\frac{\theta'}{T'}\right)^s \left\{e^{-\theta'/T'}(T'_{sb} - T') - \frac{1}{\gamma_f}\left(\frac{T'}{T'_0}\right)^s \left(\frac{a'_{f0}}{a'_f}\right)^2 \left(\frac{\mathscr{K}'}{\mathscr{K}'_0}\right) e^{-\theta'/T'_0}(T'_{0f} - T'_0)\right\}, \tag{3.15}$$

and we note the appearance of two dimensionless groups namely $(a'_f/u')^2$ of which more will be said shortly, and

$$\left(\frac{T'}{T'_0}\right)^s \left(\frac{a'_{f0}}{a'_f}\right)^2 \left(\frac{\mathscr{K}'}{\mathscr{K}'_0}\right) = \left(\frac{T'}{T'_0}\right)^{\omega+s}. \tag{3.16}$$

The final form of (3.16) follows from the fact that λ'/λ'_0 is equal to $(T'/T'_0)^\omega$, $0 < \omega \lesssim 1$, in many circumstances. Little accuracy in the modelling for present purposes will be sacrificed if we choose the simplest value,

$$\omega + s = 0, \tag{3.17}$$

and this will be adopted from now on.

The local frozen Mach number can be written as

$$\left(\frac{u'}{a'_f}\right)^2 = \frac{u'^2_i}{a^2_{f0i}} \frac{u^2}{a^2_f} \equiv M^2 \frac{u^2}{T}, \tag{3.18}$$

where

$$u \equiv \frac{u'}{u'_i}, \qquad a_f^2 \equiv \frac{a'^2_f}{a^2_{f0i}} = \frac{T'}{T'_{0i}} \equiv T. \tag{3.19}$$

The values with subscript 0i are ambient-initial values and M is the ambient initial Mach number of the flow. Since θ' is usually a large temperature (typically 2×10^4 K) it is clear that the dominant behaviour of the right-hand side of (3.15) is encapsulated in the exponentials $\exp(-\theta'/T')$ and $\exp(-\theta'/T'_0)$ in the { } term, together with the differences $(T'_{sb} - T'_0)$ and $(T'_{0f} - T'_0)$. The coefficient of { } has an important role to play as a determinant of *magnitude* of the right-hand side of (3.15), but its variations with ξ throughout the ξ-domain $(-\infty < \xi < \infty)$ that come from changes in u and T (see (3.18)) are less significant and so (3.15) will be simplified to read

$$\frac{\partial}{\partial \xi}(T_s - T_0) - \frac{\partial^2}{\partial \xi^2}(T_s - T_0) = \Lambda\left\{e^{-\theta/T}(T_{sb} - T_s) - \frac{1}{\gamma_f} e^{-\theta/T_0}(T_{0f} - T_0)\right\}, \tag{3.20}$$

where

$$\Lambda \equiv \tfrac{4}{3} c_{X0i} A_3 \theta^s \mathscr{M}^{-2}, \quad \theta \equiv \theta'/T'_{0i}, \tag{3.21}$$

and all temperatures T_s, T_0, T_{sb} and T_{0f} are now measured in units of the initial ambient value T'_{0i}. The quantity $\mathscr{M}^2$ in (3.21) is equal to M^2 at the initial instant, but must be allowed to vary (slowly) with time thereafter; that is to say one must take into account the *time*-modulations implicit in (3.18), but the spatial changes in u^2/T are neglected.

The dimensionless stagnation temperature is evidently given by

$$T_s = T + \tfrac{1}{2}(\gamma_f - 1) M^2 u^2 \tag{3.22}$$

in view of (1.11), (3.18) and (3.19).

The proposition that M^2 is very small now reduces (3.20) to a single equation for T, given that T_0 is known from the work of Section 2, as follows:

$$\underset{(C)}{\frac{\partial}{\partial\xi}(T-T_0)} - \underset{(D)}{\frac{\partial^2}{\partial\xi^2}(T-T_0)} = \Lambda\left\{\underset{(R)}{e^{-\theta/T}(T_b-T) - \frac{1}{\gamma_f}e^{-\theta/T_0}(T_{0f}-T_0)}\right\}. \tag{3.23}$$

Identification of the terms here as C (convection), D (diffusion) and R (reaction) will be useful later on.

Reverting to dimensional forms for a moment, observe that (2.11) and (3.10) combine to give

$$(1+\sigma)Q'c_F = C_v'(T_{0f}'-T_0') - C_p'(T'-T_0') + \tfrac{1}{2}(u_\infty'^2 - u'^2).$$

The final term here is of order M^2 in dimensionless variables and must be dropped to be consistent with (3.23). The temperature T_b', or T_b, is then evidently given by

$$T_b - T_0 = \frac{1}{\gamma_f}(T_{0f}-T_0). \tag{3.24}$$

Thus R in (3.23) vanishes when $T=T_0$, as it should. It will also vanish when $T=T_{b-}$, where

$$T_b - T_{b-} = \frac{1}{\gamma_f}\exp\left\{\frac{\theta}{T_{b-}} - \frac{\theta}{T_0}\right\}(T_{0f}-T_0), \tag{3.25}$$

and it is not difficult to see that this can only happen again if T_{b-} is very nearly equal to T_b, given that T_b is, say, at least 3 or 4 times larger than T_0. The fact that T_{0f} will be of this sort of size, by hypothesis, guarantees the requisite T_b behaviour, as can be seen from (3.24).

The right-hand side of (3.25) is essentially positive, hence the burnt temperature, as defined by (3.25), will be slightly less than T_b. This will be recognized by stating from now on that R in (3.23) vanishes when $T=T_0$ and $T=T_{b-}$, where the latter is found from (3.25).

Since, from (3.24),

$$T_b = \frac{1}{\gamma_f}T_{0f} + \frac{\gamma_f-1}{\gamma_f}T_0 \simeq T_{b-}, \tag{3.26}$$

and T_{0f} is a constant, as shown by (2.13), it follows that T_b and hence also T_{b-} both increase (slowly, by hypothesis) with time as T_0 does likewise.

4. Solutions

Equation (3.23) describes the behaviour of a disturbance to the unbounded ambient atmosphere that is *not* limited in amplitude (that is to say, it is not a *small* disturbance).

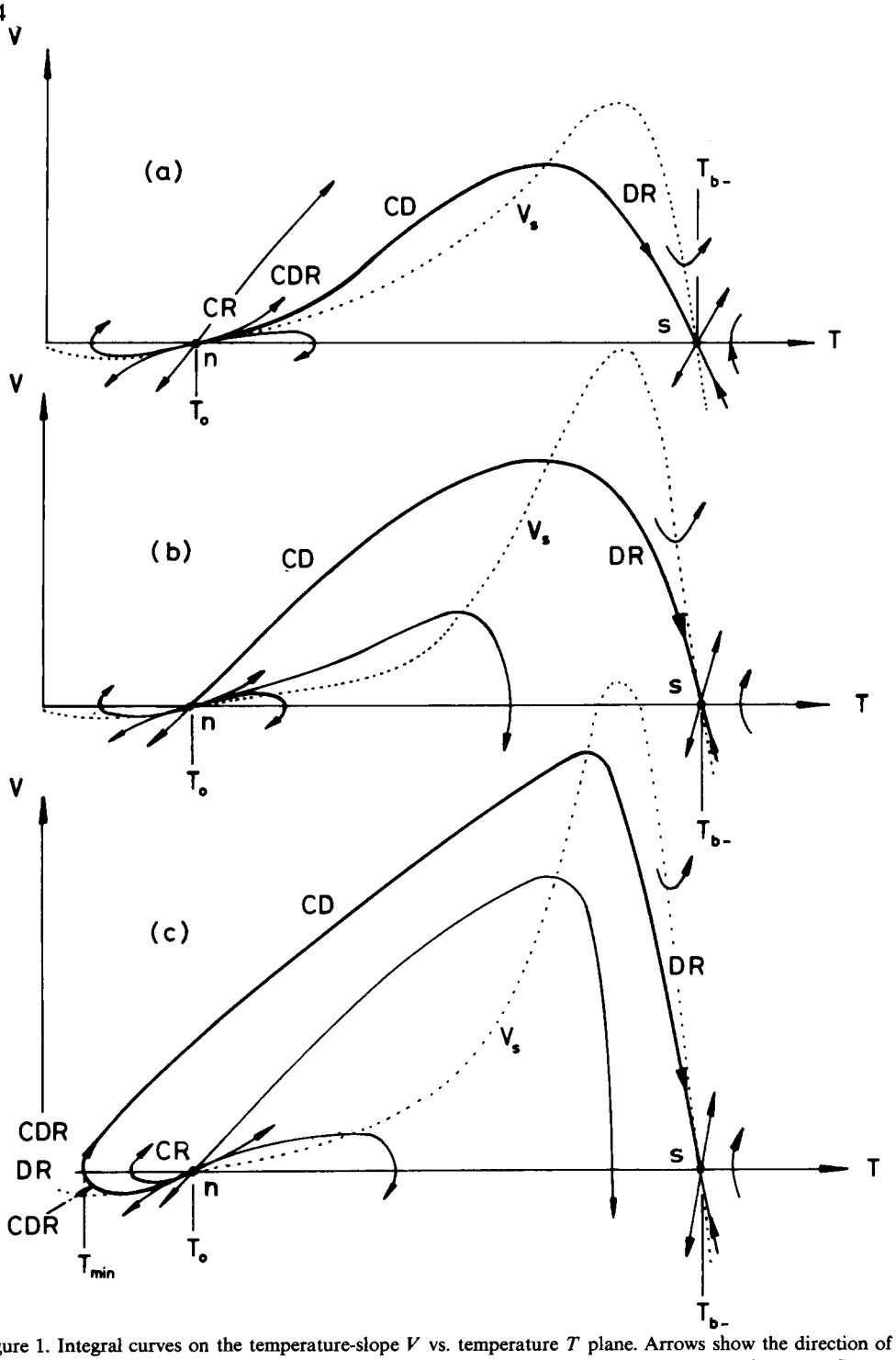

Figure 1. Integral curves on the temperature-slope V vs. temperature T plane. Arrows show the direction of ξ increasing; n is a nodal point, reached as $\xi \to -\infty$, while the saddle point at s is reached as $\xi \to +\infty$. C, D and R refer to convection, diffusion and reaction (c.f. (3.23)) and their combinations on the figures indicate the dominant local influences in the wave structure. With $\mathcal{M}_{ad}$ as the "adiabatic thermal-flame" Mach number Figures (a), (b) and (c) illustrate the cases $\mathcal{M} > \mathcal{M}_{ad}$, $\mathcal{M} = \mathcal{M}_{ad}$ and $\mathcal{M} < \mathcal{M}_{ad}$, respectively.

The disturbance is hypothesised to be travelling at a speed u'_∞ that varies only weakly with time.

First note that with

$$V \equiv \frac{\mathrm{d}(T - T_0)}{\mathrm{d}\xi} \tag{4.1}$$

(3.23) is a first-order non-linear autonomous differential equation for $V = V(T)$, namely

$$\frac{\mathrm{d}V}{\mathrm{d}T} = \frac{V - \Lambda\mathscr{C}}{V}, \tag{4.2}$$

where

$$\mathscr{C} \equiv \mathrm{e}^{-\theta/T}(T_\mathrm{b} - T) - \frac{1}{\gamma_\mathrm{f}}\mathrm{e}^{-\theta/T_0}(T_{0\mathrm{f}} - T_0). \tag{4.3}$$

Singular points of (4.2) occur when $T = T_0$ or $T_{\mathrm{b}-}$, and V is zero.

Using (3.24), $\mathscr{C}$ can be re-written as follows:

$$\mathscr{C} = \frac{1}{\gamma_\mathrm{f}}(\mathrm{e}^{-\theta/T} - \mathrm{e}^{-\theta/T_0})(T_{0\mathrm{f}} - T_0) - \mathrm{e}^{-\theta/T}(T - T_0), \tag{4.4}$$

and near to $T = T_0$ this behaves like

$$\mathscr{C} \simeq \mathrm{e}^{-\theta/T_0}\left\{\frac{\theta}{T_0^2}\frac{1}{\gamma_\mathrm{f}}(T_{0\mathrm{f}} - T_0) - 1\right\}(T - T_0) + \cdots \equiv K_0(T - T_0) + \cdots, \tag{4.5}$$

where K_0 is defined here for convenience. Thus the integral curves of (4.2) have their behaviour prescribed by

$$\frac{\mathrm{d}V}{\mathrm{d}T} = \frac{V - \Lambda K_0(T - T_0)}{V}, \tag{4.6}$$

in the neighbourhood of $T = T_0$, $V = 0$. The characteristic equation of (4.6) or, better, of the pair

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} = V - \Lambda K_0(T - T_0), \qquad \frac{\mathrm{d}T}{\mathrm{d}\xi} = V, \tag{4.7}$$

has roots equal to $\frac{1}{2} \pm (\frac{1}{4} - \Lambda K_0)^{1/2}$. Thus these roots are real, positive, and unequal since we may take it that $4\Lambda K_0$ is less than one. It follows (e.g. from [8, Chapter 4, Part B]) that the point $(T_0, 0)$ is a nodal point, that is approached only as $\xi \to -\infty$. Furthermore, all the integral curves emanate from $(T_0, 0)$ as tangents to the line

$$V = \left\{\tfrac{1}{2} - \left(\tfrac{1}{4} - \Lambda K_0\right)^{1/2}\right\}(T - T_0), \tag{4.8}$$

except for the single curve that leaves $(T_0, 0)$ along

$$V = \left\{\tfrac{1}{2} + \left(\tfrac{1}{4} - \Lambda K_0\right)^{1/2}\right\}(T - T_0). \tag{4.9}$$

In the neighbourhood of $(T_{b-}, 0)$, (4.2) behaves like

$$\frac{dV}{dT} = \frac{V + \Lambda\, e^{-\theta/T_{b-}}(T - T_{b-})}{V}, \tag{4.10}$$

whose characteristic equation has roots $1/2 \pm \{1/4 + \Lambda \exp(-\theta/T_{b-})\}^{1/2}$. These two roots are real, unequal and of different signs, so that $(T_{b-}, 0)$ is a saddle point. The integral curve that enters $(T_{b-}, 0)$ as $\xi \to \infty$ does so along the path that corresponds to the negative root; the magnitude of this root is strongly dependent on the size of $\Lambda \exp(-\theta/T_{b-})$ and from the definition of Λ in (3.21) it is clear that this is determined by the value of $\mathscr{M}^2$ for any given combustible mixture (i.e. any given θ and T_{b-}, in particular).

It is important to observe that even when $\Lambda \exp(-\theta/T_{b-})$ is large, the quantity ΛK_0 (see (4.5) for K_0) will usually be very small as a result of the dominant influence of $\exp(-\theta/T_0)$ in the latter when θ is large.

Integral-curve behaviour is sketched in Fig. 1, which will be described in more detail in the next section. The general features of this behaviour at locations other than the singular points can be explained as follows. The dotted line in Fig. 1 is the locus of points at which dV/dT vanishes, namely $V = V_s$, where

$$V_s = \Lambda \mathscr{C}$$

(see (4.2)). From the definition of $\mathscr{C}$ in (4.3) it is clear that V_s is equal to the reaction-rate term on the right-hand side of (3.23). In the present context T_0 is to be thought of as a constant; γ_f, θ and T_{0f} are essentially so. The shape of V_s is determined by the function $(T_b - T)\exp(-\theta/T)$. It can be shown that this function has a single maximum for temperatures in the interval $T_0 < T < T_{b-}$, which lies at $T_{b-} - T_{b-}^2\theta^{-1}$ to leading order for large values of θ. The integral curve of negative slope that enters $(T_{b-}, 0)$ *must* do so from the domain $0 < V < V_s$, $T < T_{b-}$. The set of integral curves that enter $(T_0, 0)$ as tangents to the line (4.8) *must lie above* V_s in the neighbourhood of $(T_0, 0)$. Any integral curve that crosses $V = 0$ at an ordinary point just do so with infinite slope. All integral curves have positive slope for $V > 0$ unless $0 < V < V_s$, $T_0 < T < T_{b-}$, when their slopes are negative.

5. Discussion of the solutions

For particular values of T_0 and T_{b-} the integral-curve behaviour depends only upon the value Λ (recall that θ, T_{0f} and γ_f are all prescribed constant values). It is sufficient for present purposes to define the propagation Mach number $\mathscr{M}$ by the following expression:

$$\mathscr{M}^{-2} = B\theta^m \exp(\theta/T_{b-}), \tag{5.1}$$

and to note that the new parameters B and m can be weak functions of time, as indeed is T_{b-}. Then (3.21) and (5.1) combine to give

$$\begin{aligned}\Lambda &= \tfrac{4}{3} c_{X0i} A_3 \theta^s B \theta^m \exp(\theta/T_{b-}) \\ &\equiv A\theta^N \exp(\theta/T_{b-}), \quad N \equiv m + s, \quad A \equiv \tfrac{4}{3} c_{X0i} A_3 B.\end{aligned} \tag{5.2}$$

Since θ can be thought of here as a largish number, in the neighbourhood of say 50 to 100,

with T_{b-} in the range of 5 to 7, for example, it follows that $\mathscr{M}^2$ will always be a small quantity (as required), when B and N are numbers of order unity. The particular choice of $\mathscr{M}^{-2}$ in (5.1) puts this Mach number in the neighbourhood of thermal-flame values and makes comparisons with the author's paper [3] easy.

The solution curve that enters the saddle point now has a slope $\frac{1}{2} - (\frac{1}{4} + A\theta^N)^{1/2}$, as can be seen from (4.10) and what follows it. When θ is in the quoted range of values and recognising that T_0 is of order unity, at least initially, it can be seen that ΛK_0 in (4.6), etc, is very small. The integral curves therefore all enter the nodal point as tangents to the line of small slope ΛK_0 (approximately), except for the single line whose slope is unity for all practical purposes. These features are illustrated in Figs. 1(a), 1(b) and 1(c), representing three different Mach numbers of propagation of the combustion wave; the node is point n and s denotes the saddle point. Furthermore, it is useful to know that, when T_b/θ is small, the maximum value of V_s is given to leading order by $A\,e^{-1}\theta^{N-1}T_b^2$. Thus as either A or N increases, which implies here an increase of B or m, the Mach number $\mathscr{M}$ decreases. Thus Fig. 1(a) represents a propagation speed that is fast relative to the value implicit in Fig. 1(b), whilst Fig. 1(c) is sketched for a speed that is similarly relatively slow.

The diffusive or D term in (3.23) is given by

$$-V\mathrm{d}V/\mathrm{d}T$$

in the terminology of Section 4, from which it follows that D is comparable with C where $|\mathrm{d}V/\mathrm{d}T|$ is of order unity. The fact that the solution sketched in Fig. 1(b) emerges from n along the singular line whose slope is equal to $+1$ for all practical purposes, coupled with the fact that the R term in (3.23) is very small in the neighbourhood of n means that C and D are the dominant process in the upstream parts of that particular flame, and Fig. 1(b) is labelled CD accordingly.

Note that R in (3.23) is equal to $\Lambda\mathscr{C} = V_s$, using (4.3) and (4.11). Thus the dotted lines in Fig. 1 give some rough and ready indication of the relative magnitude and importance of R.

Since $\mathrm{d}V/\mathrm{d}T$ is given by $\frac{1}{2} - (\frac{1}{4} + A\theta^N)^{1/2}$ near s it is numerically large for large values of θ, and N not too zero. The latter is assumed throughout Fig. 1, and accordingly all of the flame structures (implicitly) illustrated there have a predominantly diffusive-reactive balance, or "flame-sheet", in their downstream regions; the label DR on Fig. 1 denotes this fact.

Fig. 1(b) is a particular, eigenvalue-like, case for which $\mathscr{M}$ is equal to $\mathscr{M}_{ad}$ (say) that has all the features of a classical adiabatic thermal flame. In the circumstances this fact is much less remarkable than the fact that a *continuous spectrum of admissible flame speeds exists* on either side of this special value (c.f. the work of Johnson [9], where it is concluded from analysis of an equation like (3.23) that a unique eigenvalue for flame speed will only exist if the reaction-rate vanishes in a finite neighbourhood of the point $(0, T_0)$; if, as here, reaction rate or reaction-rate difference vanishes in proportion to $(T - T_0)$ as $T \to T_0$, then all speeds of propagation greater than some minimum value are possible; the existence of and conditions near to this minimum value are not dealt with in the present work, whose objectives and aims are different from Johnson's).

First consider Fig. 1(a), for which $\mathscr{M} > \mathscr{M}_{ad}$. The particular solution curve that emerges from n and enters the saddle point s does so as a tangent to the line on the V, T plane whose slope is given by

$$\tfrac{1}{2} - (\tfrac{1}{4} - \Lambda K_0)^{1/2} \simeq \Lambda K_0 \ll 1.$$

Thus D is initially a small effect, the initial structure is that of a convected-explosion or CR process. As both V and T increase on the solution curve the associated rise in dV/dT means that a CDR region appears locally and quickly blends into a largely CD domain. The DR-type of flame sheets persists at the rear of the combustion wave, and the whole fast-flame picture here is simply the unbounded space version of the faster-than-adiabatic structures described in the author's [3] analysis of half-space or burner flames. Such a system has been described as a thermal flame, CD + DR, riding at the far-downstream end of a convected-explosion precursor.

The situation represented by Fig. 1(c) has $\mathscr{M} < \mathscr{M}_{ad}$, so that the solution-curve enters s with a larger negative slope than is the case with the adiabatic flame. As a result it must emerge from n into regions of $T < T_0$ and $V < 0$; the solution-curve is still tangent to the line of small slope ΛK_0 through n. Thus the temperature far ahead of the main thermal flame (CD and DR both still exist at the rear of the wave, as shown in Fig. 1(c)) starts by decreasing to a minimum value T_{min} essentially > 0, before increasing to merge with the familiar CD-type pre-heat zone. The complex of CR, DR and CDR domains through which it makes this progress are indicated on the figure; the *un*importance of C is localised to the immediate neighbourhood of $V = 0$, of course, and one may choose to ignore the localised presence of the DR zone and treat the transition from n as essentially via domains of CR and CDR balance, before entering the pre-heat CD segment of the (non slower) thermal flame. The distinctions between Figs. 1(a) and (c) are then no so great, as the formal sequence of structures is identical in the two cases.

It is important to remember that the present quasi-steady-combustion-wave structures that formally occupy the complete space $-\infty < \xi < \infty$, *cannot do so in reality.* Since the quasi-steady solutions make $V \to 0$ as $|\xi| \to \infty$ it is clear that the time-derivative terms that have been dropped from the exact equations (3.1) and (3.2), for example, must begin to play a significant part as $|\xi|$ increases, and before V in fact vanishes. *We must therefore regard the present quasi-steady waves as entities that are not only to be properly matched into a larger field of disturbances, but which are actually strongly dependent upon the detailed nature of this surrounding disturbance field.* With the limit $\xi \to -\infty$ treated as part of a proper asymptotic matching requirement it is easy to see how the situations depicted in Figs. 1(a) and (c) will be linked with a disturbance field in which the temperature far ahead of the combustion wave is increasing above T_0 or diminishing below it, respectively. That there must be limitations to the range of acceptable disturbance fields can be appreciated; the situation is not a simple one and it is hoped to report on some illustrative cases in due course.

The previous paragraph constitutes one of the main conclusions to be drawn from the present analysis. To reiterate and expand upon its contents a little, it can now be appreciated that a truly steady-state flame is unlikely to occur in an unbounded atmosphere. However, quasi-steady structures *are* to be found, occupying the central combustion-dominated core of the whole flow field, *and of a type which exactly mimics those structures previously described for half-space or "burnder" flames* [3–5,10].

One final point: the foregoing analysis of the quasi-steady flame is put together on the proposition that the term $\mathscr{R}_0'$ must be retained in (3.5), (3.6), and hence in (3.23); as a consequence a temperature field is predicted for all ξ in $-\infty < \xi < \infty$. In view of the requirement to study flames in unbounded atmospheres that last fact is important, at least in the early part of the analysis. If the temporal variations in any particular case are in fact slow enough, then the details of conditions *near* to the node n (whose existence is consequent upon retention of $\mathscr{R}_0'$) must be important and are, of course, known. If

temporal changes are swifter than the size of $\mathscr{R}_0'$ implies, the quasi-steady structures fail in a relatively larger neighbourhood of the node, and behaviour at and near to n is then irrelevant to a solution of the complete field; the core of the quasi-steady flame structure will still be valid, of course, and will not be affected to any significant degree by the presence or absence of $\mathscr{R}_0'$.

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